Correction
Exam
Algèbra I

Solution 01:

1) Let's determine $f(A)$:

$$f(A) = \{f(x) \,/\, x \in A\} \quad 01$$

$$= \{f(0),\, f(1/2),\, f(2)\}$$

$$= \{0,\, \tfrac{2}{5}\} \quad 0{,}5$$

2) Since $f(1/2) = f(2) = \frac{2}{5}$ 01 then f is not injective.

3) Surjectivity:

f is surjective $\longleftrightarrow$ 0,5

$$\forall y \in \mathbb{R},\ \exists x \in \mathbb{R},\ y = \frac{x}{1+x^2}$$

We have:

$$y = \frac{x}{1+x^2} \rightarrow yx^2 - x + y = 0 \quad 0{,}5$$

$$\Delta = 1 - 4y^2$$

for $y = 1$, $\Delta = -3 < 0$ 0,5

$\rightarrow \nexists\, x \in \mathbb{R}$ such that $f(x) = 1$

Thus, f is not surjective on $\mathbb{R}$.

Solution 02:

Let's show that

$$\forall n \in \mathbb{N}^*: \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

a) For $n = 0$:

$$\sum_{k=0}^{0} k^2 = 0^2 = 0 \quad 0{,}5 \quad \text{and}$$

$$\frac{n(n+1)(2n+1)}{6} = \frac{0(1)(1)}{6} = 0 \quad 0{,}5$$

then $0 = 0$ 0,5

Thus $P(0)$ is true.

2) We assume that $P(n)$ is true and we show that $P(n+1)$ is true, i.e.: 01

$$\sum_{k=0}^{n+1} k^2 \overset{?}{=} \frac{(n+1)(n+2)(2n+3)}{6}$$

We have:

$$\sum_{k=0}^{n+1} k^2 = 0^2 + 1^2 + \dots + n^2 + (n+1)^2 \quad 0{,}5$$

$$= \sum_{k=0}^{n} k^2 + (n+1)^2 \quad 0{,}5$$

$$\xrightarrow{\text{by } P(n)} = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad 0{,}5$$

$$= \frac{(n+1)[n(2n+1) + (n+1)6]}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6} \quad 0{,}5$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

So $P(n+1)$ is true 0,5

Thus the relation is true.

Solution 03:

1) Let's show that R is an equivalence relation

a) **R is reflexive:**

$\forall x \in \mathbb{R}$, we have:

$x^3 - x^3 = \alpha(x^2 - x^2)$ (1)

$\rightarrow 0 = 0.$ (0,5)

Thus $x R x$ (0,5)

b) **R is symmetric:**

$\forall x, y \in \mathbb{R}$, we have

$x R y \rightarrow x^3 - y^3 = \alpha(x^2 - y^2)$ (0,5)

$\rightarrow -(y^3 - x^3) = -\alpha(y^2 - x^2)$ (0,5)

$\rightarrow y^3 - x^3 = y^2 - x^2$ (0,5)

$\rightarrow y R x$ (0,5)

c) **R is transitive:**

$\forall x, y, z \in \mathbb{R}$, we have

$(x R y) \wedge (y R z)$ (0,5)

$\rightarrow [(x^3 - y^3) = \alpha(x^2 - y^2)] \wedge [y^3 - z^3 = \alpha(y^2 - z^2)]$

$\rightarrow x^3 - y^3 + y^3 - z^3 = \alpha(x^2 - y^2 + y^2 - z^2)$

(by addition) (0,5)

$\rightarrow x^3 - z^3 = \alpha(x^2 - z^2)$ (0,5)

$\rightarrow x R z$ (0,5)

Hence R is an equivalence relation

Solution 04:

1) Let's show that $*$ is commutative

$\forall x, y \in G$, we have:

$x * y = xy + 2(x+y) + 2$

$= yx + 2(y+x) + 2$

$= y * x$ (1)

2) $(G, *)$ is a group:

$*$ is associative:

a) $*$ is associative

Let $x, y, z \in \mathbb{R}$, we have

$$(x * y) * z = (xy + 2(x+y) + 2) * z \quad 0{,}75$$

$$= (xy + 2(x+y) + 2)z + 2(xy + 2(x+y) + 2 + z) + 2$$

$$= xyz + 2xz + 2yz + 2z + 2xy + 4x + 4y + 4 + 2z + 2$$

$$= xyz + 2xy + 2xz + 2yz + 4x + 4y + 4z + 6 \; \cdots \; (1)$$

and

$$x * (y * z) = x * (yz + 2(y+z) + 2) \quad 0{,}75$$

$$= x(yz + 2(y+z) + 2) + 2(x + yz + 2(y+z) + 2) + 2$$

$$= xyz + 2xy + 2xz + 2yz + 4x + 4y + 4z + 6 \; \cdots \; \boxed{2}$$

From (1) and (2), then $*$ is associative.

b) Existence of identity element "e": 01

Let $x \in G$, We have:

$$x * e = x \rightarrow xe + 2(x+e) + 2 = x \rightarrow xe + x + 2e + 2 = 0$$

$$\rightarrow e(x+2) = -(x+2) \rightarrow e = -1 \in G$$ (since $*$ is commutative)

c) Existen of symetric element "x^{-1}":

We have: $x * x^{-1} = e \rightarrow x x^{-1} + 2(x + x^{-1}) + 2 = -1$

$$\rightarrow x^{-1}(x+2) = -2x - 3 \rightarrow x^{-1} = \frac{-2x-3}{x+2} \overset{?}{\in} G \quad 01$$

Show that $x^{-1} \in G$: 0,5

We assume that $x^{-1} = \dfrac{-2x-3}{x+2} = -2$

$$\rightarrow -2x - 3 = -2x - 4 \rightarrow -3 = -4$$ (contradiction)

Hence $(G, *)$ is a group commutative.